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New Relation between the Coefficients of a Rational Function and the intersection points with Oblique Asymptotes in a Vector Equations

Mariwan Rashid Ahmed

Charmo University, College of Medical and Applied Sciences, Department of Applied Computer, Chamchamal-Iraq
E-mail: mariwan.rashid@charmouniversity.org

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Abstract
It's clear that Vieta's formula relates the coefficients of polynomial to the sum and product of their roots. In this paper for the first time, we introduce a vector equations relate the coefficients of numerator and denominator of rational functions to the sum and product of intersection points with oblique asymptotes. Furthermore, previously we learned how to find Oblique asymptote of rational functions by Long division, but in this paper we introduce new easier method for finding oblique asymptotes of rational functions with the aid of determinant of a matrix.

Introduction

Some of the curves we have sketch in calculus have had asymptotes, that is, straight lines to which the curve draws arbitrarily close as it recedes to infinite distance from the origin. Asymptotes are of three types: vertical, horizontal, and oblique. For curves given by the graph of a function $y = f(x)$ horizontal asymptote are horizontal lines that the graph of the function approaches as x tends to positive or negative infinity. Vertical asymptotes are vertical lines near which the function grows without bounded. An oblique asymptote has a slope that is non-zero but finite, such that the graph of function approaches it as x tends to positive or negative infinity. A rational function has at most one horizontal or oblique asymptote, and possibly many vertical asymptotes. If the degree of the numerator of a rational function is one greater than the degree of the denominator, the graph has an oblique asymptote ***[1]***.

Many authors and researches use Vieta's formula to the fractional order polynomials ***[7]***, or solving polynomial equations ***[8]***. In application, growth Models with oblique asymptote were considered in ***[9]***. In this work, we present a vector equations relate the coefficients of rational functions to the sum and product of intersection points with oblique asymptotes with the aid of Vieta's theorem. Furthermore, we present a new method for finding oblique asymptotes of rational functions with the aid of determinant of a matrix.

This paper is organized as follows. Section 2 presents the necessary definitions, theorems and basic preliminaries of calculus and fundamental theorem of algebra; section 3 devoted to drive some new theorems and corollaries about oblique asymptote of rational functions; our results illustrated throughout examples in section 4. Finally section 5 includes a Conclusion for our methods.

Calculus and Fundamental Theorem of algebra

If n is a positive integer, then a polynomial of degree n over field $\mathbb{R}$ is a formal sum of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $a_i \in \mathbb{R}$ for $i = 0,1,\cdots,n, a_n \neq 0$ and x is an indeterminate variable. A rational function is any function which can be written as the ratio of two polynomial functions, where the polynomial in the denominator not equal to zero ***[2]***. The focus for this paper is a polynomial in field $\mathbb{R}[x]$.

Definition 1: *[2]*

The graph of $y = f(x)$ has a vertical asymptote at $x = a$ if either $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$ or both.

Definition 2: *[2]*

The graph of $y = f(x)$ has a horizontal asymptote $y = a$ if either $\lim_{x\to +\infty} f(x) = a$ or $\lim_{x\to -\infty} f(x) = a$ or both.

Definition 3: *[2]*

The straight line $y = ax + b$ where $a \neq 0$ is an oblique asymptote of the graph of $y = f(x)$ if either $\lim_{x\to +\infty} [f(x) - (ax + b)] = 0$ or $\lim_{x\to -\infty} [f(x) - (ax + b)] = 0$ or both.

Definition 4: *[6]*

Let $f(x)$ and $g(x)$ be a polynomials in $\mathbb{R}[x]$, with $g(x) \neq 0$. We say that $g(x)$ is a divisor or factor of $f(x)$ and write $g(x)|f(x)$, if $f(x) = g(x)q(x)$ for some polynomial $q(x)$ in $\mathbb{R}[x]$.

Theorem 1: *[3]*

A polynomial of degree n in $\mathbb{R}[x]$ can have at most n distinct roots.

Theorem 2: *[3]*

If $f(x) \in \mathbb{R}[x]$ and $f(z_0) = 0$ then $f(\overline{z_0}) = 0$; the complex roots of real polynomials come in conjugate pairs.

Theorem 3: (Fundamental Theorem of Algebra) *[4]*

If $p(x) = \sum_{i=0}^{n} a_i x^i$, $a_i \in \mathbb{R}$, $a_n \neq 0$, then there exist $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$ such that $p(x) = a_n \prod_{i=1}^{n} (x - \alpha_i)$

Theorem 4: (Vieta's formulas) *[5]*

Consider the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with degree n has n roots, call them $r_1, r_2, \cdots, r_n$. Vieta's formulas say that

$$r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n}$$

$$(r_1r_2 + r_1r_3 + \cdots + r_1r_n) + (r_2r_3 + r_2r_4 + \cdots + r_2r_n) + \cdots + r_{n-1}r_n = \frac{a_{n-2}}{a_n}$$

$$(r_1r_2r_3 + r_1r_2r_4 + \cdots + r_1r_2r_n) + (r_1r_3r_4 + r_1r_3r_5 + \cdots + r_1r_3r_n) + \cdots + r_{n-2}r_{n-1}r_n = -\frac{a_{n-3}}{a_n}$$

$$\vdots$$

$$r_1r_2r_3 \cdots r_n = (-1)^n \frac{a_0}{a_n}$$

New Theorems and Corollaries about Oblique Asymptote of Rational Functions

In this section**,** new theorems and corollaries about oblique asymptote of rational functions has been presented. We introduce a new technique for finding oblique asymptote of rational functions; we begin by deriving Mariwan's theorem for oblique asymptote.

Theorem5: (Mariwan's Theorem 1)

If $p(x)=\frac{f(x)}{g(x)}=\frac{a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+b_{m-2}x^{m-2}+\cdots+b_1x+b_0}$ is a rational function, such that $n=m+1$; $n,m\in\mathbb{Z}^+$ then the oblique asymptote of $p(x)$ is $L(x)=\frac{a_n}{b_m}x-\frac{|\Psi|}{(b_m)^2}$ where $\Psi=\begin{bmatrix}a_n & a_{n-1}\\ b_m & b_{m-1}\end{bmatrix}$, degree of $f=n$ and degree of $g=m$.

Proof:

Consider we have the following rational function

$$p(x)=\frac{f(x)}{g(x)}=\frac{a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+b_{m-2}x^{m-2}+\cdots+b_1x+b_0} \qquad (1)$$

Such that $n=m+1$and $a_i,b_j\in\mathbb{R}$, $\forall\ i=\overline{0:n}$ and $\forall\ j=\overline{0:m}$, also $a_n\neq 0$ and $b_m\neq 0$. It's clear that $p(x)$ has an oblique asymptote, since degree of the numerator of $p(x)$ is one greater than the degree of the denominator ***[1]***.

Let $L(x)=\alpha x+\beta$ be an oblique asymptote of a rational function $p(x)$, by the definition of Oblique asymptote of rational functions $p(x)-L(x)\rightarrow 0$ as x approach to infinity, it means that

$$\frac{a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+b_{m-2}x^{m-2}+\cdots+b_1x+b_0}-(\alpha x+\beta)=0 \qquad (2)$$

Now, from equation (2) we obtain

$$\alpha=\frac{a_nx^n+(a_{n-1}-\beta b_m)x^{n-1}+(a_{n-2}-\beta b_{m-1})x^{n-2}+\cdots+(a_1-\beta b_1)x+(a_0-\beta b_0)}{b_mx^{m+1}+b_{m-1}x^m+b_{m-2}x^{m-1}+\cdots+b_1x^2+b_0x} \qquad (3)$$

By dividing numerator and denominator by x^n, we obtain

$$\alpha=\frac{a_n+(a_{n-1}-\beta b_m)\frac{1}{x}+(a_{n-2}-\beta b_{m-1})\frac{1}{x^2}+\cdots+(a_1-\beta b_1)\frac{1}{x^{n-1}}+(a_0-\beta b_0)\frac{1}{x^n}}{b_m+b_{m-1}\frac{1}{x}+b_{m-2}\frac{1}{x^2}+\cdots+b_1\frac{1}{x^{n-2}}+b_0\frac{1}{x^{n-1}}} \qquad (4)$$

As x approach to infinity in equation (4), we get

$$\alpha=\frac{a_n}{b_m}$$

Substituting the value of α in equation (2) and rewriting the equation with respect to β, we get

$$\beta=\frac{a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+b_{m-2}x^{m-2}+\cdots+b_1x+b_0}-\frac{a_n}{b_m}x \qquad (5)$$

Therefore

$$\beta=\frac{(b_ma_{n-1}-a_nb_{m-1})x^m+(b_ma_{n-2}-a_nb_{m-2})x^{m-1}+\cdots+(b_ma_1-a_nb_0)x+b_ma_0}{(b_m)^2x^m+b_mb_{m-1}x^{m-1}+b_mb_{m-2}x^{m-2}+\cdots+b_mb_1x+b_mb_0} \qquad (6)$$

By dividing numerator and denominator by x^m, we obtain

$$\beta = \frac{(b_m a_{n-1} - a_n b_{m-1}) + (b_m a_{n-2} - a_n b_{m-2})\frac{1}{x} + \cdots + (b_m a_1 - a_n b_0)\frac{1}{x^{m-1}} + b_m a_0 \frac{1}{x^m}}{(b_m)^2 + b_m b_{m-1}\frac{1}{x} + b_m b_{m-2}\frac{1}{x^2} + \cdots + b_m b_1 \frac{1}{x^{m-1}} + b_m b_0 \frac{1}{x^m}} \qquad (7)$$

When x approach to infinity, from equation (7) we obtain

$$\beta = \frac{(b_m a_{n-1} - a_n b_{m-1})}{(b_m)^2} = -\frac{1}{(b_m)^2}\begin{vmatrix} a_n & a_{n-1} \\ b_m & b_{m-1} \end{vmatrix} \qquad (8)$$

Substituting the value of α and β in the function $L(x)$, we get

$$L(x) = \frac{a_n}{b_m}x - \frac{1}{(b_m)^2}|\Psi| \quad \text{, Where } \Psi = \begin{bmatrix} a_n & a_{n-1} \\ b_m & b_{m-1} \end{bmatrix}$$

From theorem (5) we obtain the following corollaries

Corollary 1: The Oblique asymptote of a rational function passes through origin if and only if $|\Psi| = 0$, where Ψ is defined in theorem (5).

Proof: (→) Suppose that The Oblique asymptote $L(x)$ as defined in theorem (5) passes through origin, this means that $L(0) = 0$, thus, we get $|\Psi| = 0$.

(←) Suppose that $|\Psi| = 0$ in $L(x)$ as defined in theorem (5), thus we get $L(0) = 0$, this means that $L(x)$ pass through origin.

Corollary 2: Every oblique asymptotes of rational function $p(x)$ intersect x-axis at the point $\left(\frac{|\Psi|}{a_n b_m}, 0\right)$ and intersect y-axis at the point $\left(0, -\frac{|\Psi|}{(b_m)^2}\right)$

Proof: To determine the x- intercept of oblique asymptote $L(x)$ as defined in theorem (5), we set $y = 0$ and solve for x we get $x = \frac{|\Psi|}{a_n b_m}$. Similarly, to determine the y- intercept of oblique asymptote $L(x)$, we set $x = 0$, and solve for y we get $y = -\frac{|\Psi|}{(b_m)^2}$.

Theorem 6: (Mariwan's Theorem 2)

The number of intersection point of a rational function $p(x)$ in equation (1) with its oblique asymptotes at most $n - 2$.

Proof: To find the intersection points of rational function $p(x)$ and oblique asymptote $L(x)$ as defined in theorem (5) we put $p(x) = L(x)$, thus, we obtain

$$\frac{f(x)}{g(x)} = \frac{a_n}{b_m}x - \frac{1}{(b_m)^2}|\Psi| \qquad (9)$$

Multiplying both sides of equation (9) by $g(x)(b_m)^2$ we get

$$(b_m)^2 f(x) - a_n b_m x g(x) + g(x)|\Psi| = 0$$

Expanding above equation and after some minor algebraic calculation we obtain

$$(b_m^2\, a_{n-2} - a_n b_m b_{m-2} + |\Psi| b_{m-1})x^{n-2} + (b_m^2\, a_{n-3} - a_n b_m b_{m-3} + |\Psi| b_{m-2})x^{n-3} + \cdots$$

$$+(b_m^2\, a_2 - a_n b_m b_1 + |\Psi| b_2)x^2 + (b_m^2\, a_1 - a_n b_m b_0 + |\Psi| b_1)x + (b_m^2\, a_0 + |\Psi| b_0) = 0$$

This equation can be expressed in the following form

$$\bar{R}(x) = \sum_{i=2}^{n-1} \psi_i \, x^{n-i} + \psi_n = 0 \tag{10}$$

Where

$$\left.\begin{aligned} \psi_i &= b_m^2 \, a_{n-i} - a_n b_m b_{m-i} + |\Psi| b_{m-i+1} \,, \forall \, i = 2,3, \dots, n-1 \\ &= |\Psi| b_{m-i+1} - |\bar{\Psi}_i| b_m \,, \text{when } \bar{\Psi}_i = \begin{bmatrix} a_n & a_{n-i} \\ b_m & b_{m-i} \end{bmatrix} \,, \forall \, i = 2,3, \dots, n-1 \\ \psi_n &= a_0 b_m^2 + |\Psi| b_0 \end{aligned}\right\} \tag{11}$$

Since $\bar{R}(x)$ in equation (10) is a polynomial degree at most $n-2$, so according to the theorem (1), the equation (10) has at most $n-2$ roots, the proof is complete.

Corollary 3: The remainder of rational function $p(x)$ as defined in equation (1) is $R(x) = \frac{1}{(b_m)^2} \bar{R}(x)$ where $\bar{R}(x)$ defined in equation (10).

Proof: From the proof of theorem (6), we note that $\bar{R}(x) = g(x)(b_m)^2 \left[\frac{f(x)}{g(x)} - L(x)\right]$

Now, according to the division theorem for polynomials $\frac{f(x)}{g(x)} = L(x) + \frac{R(x)}{g(x)}$ where $R(x)$ is a remainder polynomial, this implies that, the remainder of the rational function $p(x)$ as defined in equation (1) is $R(x) = g(x) \left[\frac{f(x)}{g(x)} - L(x)\right]$, hence we get $R(x) = \frac{1}{(b_m)^2} \bar{R}(x)$

Corollary 4: The real roots of the remainder polynomials $R(x) = \frac{1}{(b_m)^2} (\sum_{i=2}^{n-1} \psi_i \, x^{n-i} + \psi_n)$, Where ψ_i and ψ_n, $\forall \, i = 2,3, \dots, n-1$ defined in equation (11) are intersection points between rational function $p(x)$ in equation (1) and oblique asymptote's $L(x)$, such that denominator of $p(x)$ not equal to zero at these roots.

Proof: From the division theorem of polynomials we have $p(x) = L(x) + \frac{R(x)}{g(x)}$, where $p(x), L(x)$ defined in theorem (5) and $g(x) \neq 0$. If $R(x) = 0$ we get $p(x) = L(x)$, this means that the points of intersection of $p(x)$ and $L(x)$ are the real roots of the remainder polynomial $R(x)$.

Corollary 5: If $\psi_i = 0, \forall \, i = 2,3, \dots, n-1$ but $\psi_n \neq 0$ in equation (11) then $p(x)$ never intersects oblique asymptotes.

Proof: If $\psi_i = 0, \forall \, i = 2,3, \dots, n-1$ and $\psi_n \neq 0$, then from corollary (3), we get $R(x) = \frac{1}{(b_m)^2} \psi_n$, but ψ_n is a non-zero constant, hence $R(x) \neq 0$, this means that $R(x)$ has not any root. Hence from corollary (4) we conclude that $p(x)$ and $L(x)$ have not any intersection points.

Corollary 6: If $n = 2$ in equation (1) and $\psi_2 \neq 0$ then the rational function $p(x)$ never intersects oblique asymptotes.

Proof: If $n = 2$ then from corollary (3) we get $R(x) = \frac{1}{(b_1)^2} \psi_2$, but $\psi_2 \neq 0$, hence the remainder $R(x)$ is a non-zero constant $R(x) \neq 0$, this means that $R(x)$ has not any real roots. Hence from corollary (4) we conclude that $p(x)$ and $L(x)$ have not any intersection points.

Theorem 7: (Mariwan's Theorem 3)

The number of intersection points of a rational function $p(x)$ in equation (1) with oblique asymptotes $L(x)$ at most k, when $k = max\{n - i \mid \psi_i \neq 0, \forall \, i = \overline{2 : n-1}\}$, and $\psi_i, \forall \, i = \overline{2 : n-1}$ are defined in equation (11).

Proof: let $k = max\{n - i \mid \psi_i \neq 0, \forall \, i = \overline{2 : n-1}\}$

Therefore, the remainder $R(x) = \frac{1}{(b_m)^2}\left(\sum_{i=2}^{n-1}\psi_i\, x^{n-i} + \psi_n\right)$ is a polynomial of degree k, hence according to the theorem (1) the remainder $R(x)$ has at most k roots, but in corollary (4) we have, the real roots of the remainder polynomials $R(x)$ are the intersection points between rational function $p(x)$ and oblique asymptote's $L(x)$, the proof is complete.

Theorem 8: (Mariwan's theorem 4)

If $\lambda_1, \lambda_2, \dots, \lambda_{n-2}$ are exactly $n-2$ intersection points of a rational function $p(x)$ in equation (1) with its oblique asymptote's $L(x)$ where $n \geq 3$, then Mariwan's vector equation is

$$\vec{a} = \Omega\,\vec{r} - \frac{|\Psi|}{(b_m)^2}\,\vec{b} + \frac{a_n}{b_m}\,\vec{c} \tag{12}$$

Where $\vec{a} = (a_{n-2}, a_{n-3}, \cdots, a_1, a_0\ )^T$, $\vec{b} = (b_{n-2}, b_{n-3}, \cdots, b_1, b_0\ )^T$, $\vec{c} = (b_{n-3}, b_{n-4}, \cdots, b_0, 0\ )^T$

$$\vec{r} = \left(1, -\sum_{1\leq i\leq n-2}\lambda_i\,, +\sum_{1\leq i<j\leq n-2}\lambda_i\lambda_j\,, \cdots, (-1)^{n-3}\left(\prod_{\substack{i=1\\ i\neq 1}}^{n-2}\lambda_i + \prod_{\substack{i=1\\ i\neq 2}}^{n-2}\lambda_i + \cdots + \prod_{\substack{i=1\\ i\neq n-2}}^{n-2}\lambda_i\right), (-1)^{n-2}\prod_{i=1}^{n-2}\lambda_i\ \right)^T$$

$$\Omega = \frac{b_{m-1}}{{b_m}^2}|\Psi| - \frac{|\Phi|}{b_m}\ , \Psi = \begin{bmatrix} a_n & a_{n-1} \\ b_m & b_{m-1}\end{bmatrix} \text{ and } \Phi = \begin{bmatrix} a_n & a_{n-2} \\ b_m & b_{m-2}\end{bmatrix}$$

Proof: Suppose

$$q(x) = \Omega(x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_{n-2})\,,\ \Omega \in \mathbb{R} \tag{13}$$

Be a polynomial's have real root's $\lambda_1, \lambda_2, \dots, \lambda_{n-2}$. Now by equating $R(x)$ in corollary (4) and $q(x)$ in equation (13) we get

$$\frac{1}{(b_m)^2}\left(\sum_{i=2}^{n-1}\psi_i\, x^{n-i} + \psi_n\right) = \Omega(x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_{n-2}) \tag{14}$$

Where ψ_i and ψ_n , $\forall\ i = \overline{2, n-1}$ defined in equation (11). Now by using Vieta's theorem, equation (14) becomes

$$\frac{1}{(b_m)^2}\ \left(\sum_{i=2}^{n-1}\psi_i\, x^{n-i} + \psi_n\right) = \Omega x^{n-2} + \varphi_1 x^{n-3} + \varphi_2 x^{n-4} + \cdots + \varphi_{n-4}x^2 + \varphi_{n-3}x + \varphi_{n-2} \tag{15}$$

Where

$$\left.\begin{array}{l} \varphi_1 = -\,\Omega\sum_{1\leq i\leq n-2}\lambda_i \\ \varphi_2 = +\Omega\sum_{1\leq i<j\leq n-2}\lambda_i\lambda_j \\ \qquad\vdots \\ \varphi_{n-3} = (-1)^{n-3}\Omega\left(\prod_{\substack{i=1\\ i\neq 1}}^{n-2}\lambda_i + \prod_{\substack{i=1\\ i\neq 2}}^{n-2}\lambda_i + \cdots + \prod_{\substack{i=1\\ i\neq n-2}}^{n-2}\lambda_i\right) \\ \varphi_{n-2} = (-1)^{n-2}\Omega\ \prod_{i=1}^{n-2}\lambda_i \end{array}\right\} \tag{16}$$

By equating the leading coefficients of both sides in equation (15) we obtain $\frac{1}{(b_m)^2}\psi_2 = \Omega$, but from equation (11) we have $\psi_2 = b_m^2 a_{n-2} - a_n\, b_m b_{m-2} + |\Psi| b_{m-1}$, thus we get $a_{n-2} - \frac{a_n b_{m-2}}{b_m} + |\Psi|\frac{b_{m-1}}{(b_m)^2} = \Omega$. Similarly, by equating the other corresponding coefficients of both sides in equation (15) and by using equation (11), we obtain the following system of equations

$$\left.\begin{aligned}
\frac{1}{(b_m)^2}\psi_2 &= a_{n-2} - \frac{a_n b_{m-2}}{b_m} + |\Psi|\frac{b_{m-1}}{(b_m)^2} = \Omega \\
\frac{1}{(b_m)^2}\psi_3 &= a_{n-3} - \frac{a_n b_{m-3}}{b_m} + |\Psi|\frac{b_{m-2}}{(b_m)^2} = -\,\Omega\sum_{1\le i\le n-2}\lambda_i \\
\frac{1}{(b_m)^2}\psi_4 &= a_{n-4} - \frac{a_n b_{m-4}}{b_m} + |\Psi|\frac{b_{m-3}}{(b_m)^2} = +\Omega\sum_{1\le i<j\le n-2}\lambda_i\lambda_j \\
&\vdots \\
\frac{1}{(b_m)^2}\psi_{n-1} &= a_1 - \frac{a_n b_0}{b_m} + |\Psi|\frac{b_1}{(b_m)^2} = \Omega(-1)^{n-3}\left(\prod_{\substack{i=1\\ i\neq 1}}^{n-2}\lambda_i + \prod_{\substack{i=1\\ i\neq 2}}^{n-2}\lambda_i + \cdots + \prod_{\substack{i=1\\ i\neq n-2}}^{n-2}\lambda_i\right) \\
\frac{1}{(b_m)^2}\psi_n &= a_0 + |\Psi|\frac{b_0}{(b_m)^2} = \Omega(-1)^{n-2}\prod_{i=1}^{n-2}\lambda_i
\end{aligned}\right\} \tag{17}$$

Now, from the first equation in system of equations (17) we get

$$\Omega = \frac{b_{m-1}}{(b_m)^2}|\Psi| - \frac{|\Phi|}{b_m} \tag{18}$$

Where $\Psi = \begin{bmatrix} a_n & a_{n-1} \\ b_m & b_{m-1}\end{bmatrix}$ And $\Phi = \begin{bmatrix} a_n & a_{n-2} \\ b_m & b_{m-2}\end{bmatrix}$

We can write above system of equations (17) as the form vector equations as follows

$$\begin{bmatrix} a_{n-2} \\ a_{n-3} \\ a_{n-4} \\ a_{n-5} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix} = \Omega \begin{bmatrix} 1 \\ -\sum_{1\le i\le n-2}\lambda_i \\ +\sum_{1\le i<j\le n-2}\lambda_i\lambda_j \\ -\sum_{1\le i<j<k\le n-2}\lambda_i\lambda_j\lambda_k \\ \vdots \\ (-1)^{n-3}\left(\prod_{\substack{i=1\\ i\neq 1}}^{n-2}\lambda_i + \prod_{\substack{i=1\\ i\neq 2}}^{n-2}\lambda_i + \cdots + \prod_{\substack{i=1\\ i\neq n-2}}^{n-2}\lambda_i\right) \\ (-1)^{n-2}\prod_{i=1}^{n-2}\lambda_i \end{bmatrix} - \frac{|\Psi|}{(b_m)^2}\begin{bmatrix} b_{n-2} \\ b_{n-3} \\ b_{n-4} \\ b_{n-5} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix} + \frac{a_n}{b_m}\begin{bmatrix} b_{n-3} \\ b_{n-4} \\ b_{n-5} \\ b_{n-6} \\ \vdots \\ b_0 \\ 0 \end{bmatrix}$$

Thus, we obtain Mariwan's vector equation (12).

Theorem 9: (Mariwan's theorem5)

If the equation $p(x) - L(x) = 0$ have only $n - n^*$ roots $\lambda_1, \lambda_2, \dots, \lambda_{n-n^*} \in \mathbb{C}$, when $n^* \in \{2,3,\dots,n-1\}$ and $n \ge 3$, $p(x)$ and $L(x)$ defined in theorem (5), then Mariwan's vector equation is

$$\overrightarrow{a} = \Omega\,\overrightarrow{r} - \frac{|\Psi|}{(b_m)^2}\,\overrightarrow{b} + \frac{a_n}{b_m}\,\overrightarrow{c} \tag{19}$$

Where $\overrightarrow{a} = (a_{n-2}, a_{n-3}, \cdots, a_1, a_0)^T$, $\overrightarrow{b} = (b_{n-2}, b_{n-3}, \cdots, b_1, b_0)^T$, $\overrightarrow{c} = (b_{n-3}, b_{n-4}, \cdots, b_0, 0)^T$

$\overrightarrow{r} = (\underbrace{0,0,\cdots,0}_{(n^*-2)\text{ times}},1,\omega_1,\omega_2,\cdots,\omega_{n-n^*})^T$, $\Omega = \frac{b_{m-n^*+1}}{(b_m)^2}|\Psi| - \frac{|\Phi|}{b_m}$, $\Psi = \begin{bmatrix} a_n & a_{n-1} \\ b_m & b_{m-1}\end{bmatrix}$ and $\Phi = \begin{bmatrix} a_n & a_{n-n^*} \\ b_m & b_{m-n^*}\end{bmatrix}$

$\omega_1 = -\sum_{1\le i\le n-n^*}\lambda_i$, $\omega_2 = +\sum_{1\le i<j\le n-n^*}\lambda_i\lambda_j$, $\omega_3 = -\sum_{1\le i<j<k\le n-n^*}\lambda_i\lambda_j\lambda_k$

$\omega_{n-n^*-1} = (-1)^{n-n^*-1}\left(\prod_{\substack{i=1\\ i\neq 1}}^{n-n^*}\lambda_i + \prod_{\substack{i=1\\ i\neq 2}}^{n-n^*}\lambda_i + \cdots + \prod_{\substack{i=1\\ i\neq n-n^*}}^{n-n^*}\lambda_i\right)$, $\omega_{n-n^*} = (-1)^{n-n^*}\prod_{i=1}^{n-n^*}\lambda_i$

Proof: Suppose

$$q(x) = \Omega(x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_{n-n^*}) \tag{20}$$

Be a polynomial's have root's $\lambda_1, \lambda_2, \dots, \lambda_{n-n^*} \in \mathbb{C}$, where $n^* \in \{2,3,\dots,n-1\}$ and $n \ge 3$. Now, by equating the remainder polynomial $R(x)$ in corollary (3) and $q(x)$ in equation (20) we get

$$\frac{1}{(b_m)^2}\left(\sum_{i=2}^{n-1}\psi_i\, x^{n-i}+\psi_n\right)=\Omega(x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_{n-n^*}) \tag{21}$$

Where ψ_i and ψ_n , $\forall\, i=\overline{2,n-1}$ defined in equation (11). Now by using Vieta's theorem, equation (21) becomes

$$\frac{1}{(b_m)^2}\left(\sum_{i=2}^{n-1}\psi_i\, x^{n-i}+\psi_n\right)=\Omega x^{n-n^*}+\varphi_1 x^{n-n^*-1}+\varphi_2 x^{n-n^*-2}+\dots+\varphi_{n-n^*-2}x^2+\varphi_{n-n^*-1}x+\varphi_{n-n^*} \tag{22}$$

Where $\varphi_1=-\,\Omega\sum_{1\le i\le n-n^*}\lambda_i$

$\varphi_2=+\Omega\sum_{1\le i<j\le n-n^*}\lambda_i\lambda_j$

$\vdots$

$\varphi_{n-n^*-1}=(-1)^{n-n^*-1}\Omega\left(\prod_{\substack{i=1\\ i\ne 1}}^{n-n^*}\lambda_i+\prod_{\substack{i=1\\ i\ne 2}}^{n-n^*}\lambda_i+\dots+\prod_{\substack{i=1\\ i\ne n-n^*}}^{n-n^*}\lambda_i\right)$

$\varphi_{n-n^*}=(-1)^{n-n^*}\,\Omega\prod_{i=1}^{n-n^*}\lambda_i$

Now, by equating the corresponding coefficients of both sides in equation (22), we obtain $\frac{1}{(b_m)^2}\psi_i=0, \forall\, i=\overline{2{:}\,n^*-1}$ but from equation (11), we have $\psi_i=b_m^2\,a_{n-i}-a_nb_mb_{m-i}+|\Psi|b_{m-i+1}$, $\forall\, i=\overline{2{:}\,n^*-1}$ thus we get

$a_{n-i}-\frac{a_nb_{m-i}}{b_m}+|\Psi|\frac{b_{m-i+1}}{(b_m)^2}=0$, $\forall\, i=\overline{2{:}\,n^*-1}$. Similarly, by equating the coefficients of x^{n-n^*} of both sides in equation (22), we get $\frac{1}{(b_m)^2}\psi_{n^*}=\Omega$, but from equation (11), we have $\psi_{n^*}=b_m^2\,a_{n-n^*}-a_nb_mb_{m-n^*}+|\Psi|b_{m-n^*+1}$ thus we get $a_{n-n^*}-\frac{a_nb_{m-n^*}}{b_m}+|\Psi|\frac{b_{m-n^*+1}}{(b_m)^2}=\Omega$. Similarly, by equating the other corresponding coefficients in equation (22), we obtain the following system of equations:

$$\left.\begin{array}{l}
\frac{1}{(b_m)^2}\psi_i=a_{n-i}-\frac{a_nb_{m-i}}{b_m}+|\Psi|\frac{b_{m-i+1}}{(b_m)^2}=0\ ,\forall\, i=2,3,\dots,n^*-1\\
\frac{1}{(b_m)^2}\psi_{n^*}=a_{n-n^*}-\frac{a_nb_{m-n^*}}{b_m}+|\Psi|\frac{b_{m-n^*+1}}{(b_m)^2}=\Omega\\
\frac{1}{(b_m)^2}\psi_{n^*+1}=a_{n-n^*-1}-\frac{a_nb_{m-n^*-1}}{b_m}+|\Psi|\frac{b_{m-n^*}}{(b_m)^2}=-\Omega\sum_{1\le i\le n-n^*}\lambda_i\\
\frac{1}{(b_m)^2}\psi_{n^*+2}=a_{n-n^*-2}-\frac{a_nb_{m-n^*-2}}{b_m}+|\Psi|\frac{b_{m-n^*-1}}{(b_m)^2}=+\Omega\sum_{1\le i<j\le n-n^*}\lambda_i\lambda_j\\
\qquad\vdots\\
\frac{1}{(b_m)^2}\psi_{n-1}=a_1-\frac{a_nb_0}{b_m}+|\Psi|\frac{b_1}{(b_m)^2}=(-1)^{n-n^*-1}\Omega\left(\prod_{\substack{i=1\\ i\ne 1}}^{n-n^*}\lambda_i+\prod_{\substack{i=1\\ i\ne 2}}^{n-n^*}\lambda_i+\dots+\prod_{\substack{i=1\\ i\ne n-n^*}}^{n-n^*}\lambda_i\right)\\
\frac{1}{(b_m)^2}\psi_n=a_0+|\Psi|\frac{b_0}{(b_m)^2}=(-1)^{n-n^*}\Omega\prod_{i=1}^{n-n^*}\lambda_i
\end{array}\right\} \tag{23}$$

From the second equation in system of equations (23), we get

$\Omega=\frac{b_{m-n^*+1}}{(b_m)^2}|\Psi|-\frac{|\Phi|}{b_m}$, where $\Psi=\begin{bmatrix}a_n & a_{n-1}\\ b_m & b_{m-1}\end{bmatrix}$ and $\Phi=\begin{bmatrix}a_n & a_{n-n^*}\\ b_m & b_{m-n^*}\end{bmatrix}$

We can write above system of equations (23) as the form vector equation as follows

$$\begin{bmatrix} a_{n-2} \\ a_{n-3} \\ \vdots \\ a_{n-n^*+1} \\ a_{n-n^*} \\ a_{n-n^*-1} \\ a_{n-n^*-2} \\ a_{n-n^*-3} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix} = \Omega \begin{bmatrix} \left.\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} (n^*-2) \text{ times} \\ 1 \\ -\sum_{1\le i\le n-n^*} \lambda_i \\ +\sum_{1\le i<j\le n-n^*} \lambda_i\lambda_j \\ -\sum_{1\le i<j<k\le n-n^*} \lambda_i\lambda_j\lambda_k \\ \vdots \\ (-1)^{n-n^*-1}\left(\prod_{\substack{i=1\\ i\neq 1}}^{n-n^*} \lambda_i + \prod_{\substack{i=1\\ i\neq 2}}^{n-n^*} \lambda_i + \cdots + \prod_{\substack{i=1\\ i\neq n-n^*}}^{n-n^*} \lambda_i\right) \\ (-1)^{n-n^*} \prod_{i=1}^{n-n^*} \lambda_i \end{bmatrix} - \frac{|\Psi|}{(b_m)^2} \begin{bmatrix} b_{n-2} \\ b_{n-3} \\ \vdots \\ b_{n-n^*+1} \\ b_{n-n^*} \\ b_{n-n^*-1} \\ b_{n-n^*-2} \\ b_{n-n^*-3} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix} + \frac{a_n}{b_m} \begin{bmatrix} b_{n-3} \\ b_{n-4} \\ \vdots \\ b_{n-n^*} \\ b_{n-n^*-1} \\ b_{n-n^*-2} \\ b_{n-n^*-3} \\ b_{n-n^*-4} \\ \vdots \\ b_0 \\ 0 \end{bmatrix} \quad (24)$$

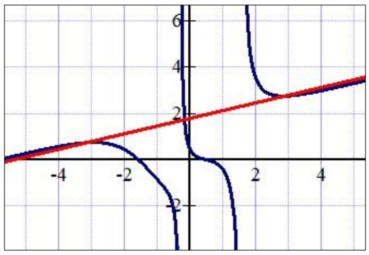
Thus, we obtain Mariwan's vector equation (19), the proof is complete.

As a special case, if the equation $p(x) - L(x) = 0$ has not any roots i.e. $n^* = n$, in this case we put $\Omega = \frac{a_0(b_m)^2 + b_0|\Psi|}{(b_m)^2}$ this means that, the remainder is a non-zero constant. Also if the equation $p(x) - L(x) = 0$ has infinite roots we put $\Omega = 0$ in vector equation (19), in this case the equation (19) becomes $\vec{a} = -\frac{|\Psi|}{(b_m)^2}\vec{b} + \frac{a_n}{b_m}\vec{c}$.

Illustrative Examples:

In this section, in order to show the efficiency of our proposed methods, we shall illustrate our theorems by some examples.

Example 1: We first consider the rational function $p(x) = \frac{\frac{1}{2}x^5 + 3x^4 - 3x^2 + 3x - 1}{\frac{3}{2}x^4 + x^3 - 7x - 2}$, in this example we have $n = 5$, $m = 4$, $a_5 = \frac{1}{2}$, $a_4 = 3$, $b_4 = \frac{3}{2}$ and $b_3 = 1$.



From Mariwan's Theorem1 we have $\Psi = \begin{bmatrix} \frac{1}{2} & 3 \\ \frac{3}{2} & 1 \end{bmatrix}$

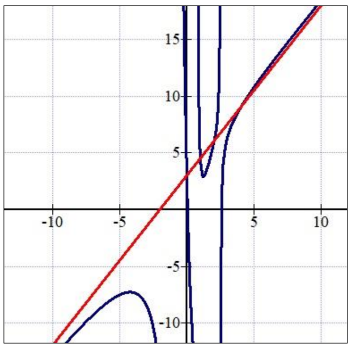
Hence the oblique asymptote of $p(x)$ is $L(x) = \frac{a_5}{b_4}x - \frac{|\Psi|}{(b_4)^2} = \frac{1}{3}x + \frac{16}{9}$

Example 2: Consider we have the rational function $p(x) = \frac{3x^6 - 2x^5 + x^4 + a_3x^3 + a_2x^2 + a_1x + 7}{2x^5 + b_4x^4 + x^3 - 2x + b_0}$ where $n = 6$, $m = 5$, $a_6 = 3$, $a_5 = -2$, $a_4 = 1$, $a_0 = 7$, $b_5 = 2$, $b_3 = 1$, $b_2 = 0$, $b_1 = -2$ and the coefficients a_1, a_2, a_3, b_0 and b_4 are unknown. If we know the rational functions $p(x)$ intersect oblique asymptotes at the point $x = 0, x = 1, x = 2, x = 4$ and the oblique asymptote intersect x-axis at the point $x = -2$.

Now, from corollary 2 we obtain $\frac{|\Psi|}{a_6 b_5} = -2$ thus we get $|\Psi| = -12$. From Mariwan's theorem 4, we have

$|\Psi| = \begin{vmatrix} 3 & -2 \\ 2 & b_4 \end{vmatrix}$ This implies that $b_4 = -\frac{16}{3}$ and also $|\Phi| = \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 1$, and $\Omega = \frac{b_4}{(b_5)^2}|\Psi| - \frac{|\Phi|}{b_5} = \frac{31}{2}$

Now, we applying equation 12 we get



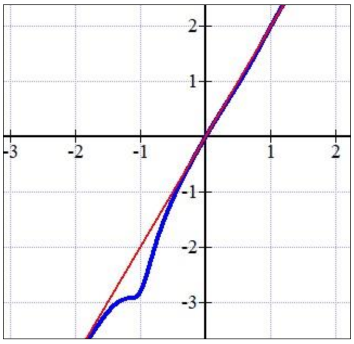
$$\begin{bmatrix} 1 \\ a_3 \\ a_2 \\ a_1 \\ 7 \end{bmatrix} = \frac{31}{2}\begin{bmatrix} 1 \\ -7 \\ 14 \\ -8 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -\frac{16}{3} \\ 1 \\ 0 \\ -2 \\ b_0 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ 0 \\ -2 \\ b_0 \\ 0 \end{bmatrix}$$

Hence $a_3 = -\frac{211}{2}$, $a_2 = 214$, $a_1 = -\frac{253}{2}$ and $b_0 = \frac{7}{3}$

Example 3: Consider we have the rational function $p(x)=\frac{a_9x^9-2x^8+a_6x^6-3x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}{5x^8-x^7+b_6x^6+6x^5-2x^4+3x^2-x+3}$ where $n=9$, $m=8$, $a_8=-2$, $a_7=0$, $a_5=-3$, $b_8=5$, $b_7=-1$, $b_5=6$, $b_4=-2$, $b_3=0$, $b_2=3$, $b_1=-1$, $b_0=3$ and the coefficients $a_0, a_1, a_2, a_3, a_4, a_6, a_9$ and b_6 are unknown. If we know the oblique asymptotes pass through origin and the roots of $p(x)-L(x)=0$ are $x=0$ multiplicity two $x=1, x=i, x=-i$.

Now, from corollary (1) we obtain $|\Psi|=\begin{vmatrix} a_9 & -2 \\ 5 & -1 \end{vmatrix}=0$, this implies that $a_9=10$. From Mariwan's theorem5, we have

$\Phi=\begin{bmatrix} 10 & -3 \\ 5 & -2 \end{bmatrix}$, and $\Omega=\frac{b_5}{b_8^2}|\Psi|-\frac{1}{b_8}|\Phi|=1$.Now from equation (19) we get



$$\begin{bmatrix} 0 \\ a_6 \\ -3 \\ a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}+2\begin{bmatrix} b_6 \\ 6 \\ -2 \\ 0 \\ 3 \\ -1 \\ 3 \\ 0 \end{bmatrix}$$

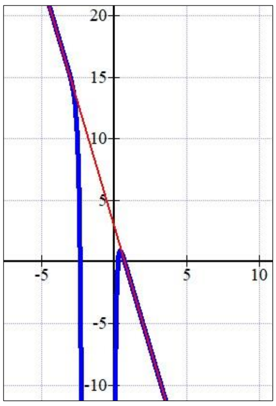
This implies that $a_0=0, a_1=6, a_2=-3, a_3=7, a_4=-1, a_6=12, a_9=10$ and $b_6=0$

Example 4: Consider we have $p(x)=\frac{a_{11}x^{11}-74x^{10}+a_9x^9-690x^8+a_7x^7-756x^6-174x^5+a_4x^4+366x^3+a_2x^2+172x-66}{2x^{10}+20x^9+b_8x^8+240x^7+b_6x^6+504x^5+420x^4+b_3x^3+90x^2+b_1x+b_0}$ where $n=11, m=10, a_{10}=-74, a_8=-690, a_6=-756, a_5=-174, a_3=366, a_1=172, a_0=-66, b_{10}=2, b_9=20, b_7=240, b_5=504, b_4=420, b_2=90$ and the coefficients $a_2, a_4, a_7, a_9, a_{11}, b_0, b_1, b_3, b_6,$ and b_8 are unknown coefficients. If we know the oblique asymptotes pass through $\left(\frac{-1}{4},4\right)$and the roots of $p(x)-L(x)=0$ are $x=1$ multiplicity two $x=-3$ and $x=\pm 2i$.

Now, from Mariwan's theorem1we have $L(x)=\frac{a_{11}}{b_{10}}x-\frac{|\Psi|}{(b_{10})^2}=\frac{a_{11}}{2}x-\frac{\begin{vmatrix} a_{11} & -74 \\ 2 & 20 \end{vmatrix}}{4}$, but since the oblique asymptotes pass through $\left(\frac{-1}{4},4\right)$, hence $L\left(\frac{-1}{4}\right)=4$ this implies that $a_{11}=-8$. Also from Mariwan's theorem5 we have

$|\Psi|=\begin{vmatrix} -8 & -74 \\ 2 & 20 \end{vmatrix}=-12, |\Phi|=\begin{vmatrix} -8 & -174 \\ 2 & 420 \end{vmatrix}=-3012$, and $\Omega=\frac{b_5}{(b_{10})^2}|\Psi|-\frac{1}{b_m}|\Phi|=-6$

Now, we applying equation (19) we get



$$\begin{bmatrix} a_9 \\ -690 \\ a_7 \\ -756 \\ -174 \\ a_4 \\ 366 \\ a_2 \\ 172 \\ -66 \end{bmatrix}=-6\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ 7 \\ -20 \\ 12 \end{bmatrix}+3\begin{bmatrix} 20 \\ b_8 \\ 240 \\ b_6 \\ 504 \\ 420 \\ b_3 \\ 90 \\ b_1 \\ b_0 \end{bmatrix}-4\begin{bmatrix} b_8 \\ 240 \\ b_6 \\ 504 \\ 420 \\ b_3 \\ 90 \\ b_1 \\ b_0 \\ 0 \end{bmatrix}$$

This implies that $a_2=148, a_4=294, a_7=-960, a_9=-300, a_{11}=-8, b_0=2, b_1=20, b_3=240, b_6=420$, and $b_8=90$

Example 5: Consider we have $p(x)=\frac{6x^6+2x^5+a_4x^4+a_3x^3+\frac{17}{4}x^2-12x+\frac{77}{8}}{4x^5+2x^4-3x^2+x+\frac{3}{2}}$ where $n=6, m=5, a_6=6, a_5=2, a_2=\frac{17}{4}$, $a_1=-12, a_0=\frac{77}{8}, b_5=4, b_4=2, b_3=0, b_2=-3, b_1=1, b_0=\frac{3}{2}$ and the coefficients a_3 and a_4 are unknown. If we know, the function $p(x)-L(x)$ has only two roots λ_1 and λ_2. To determine the value of $\lambda_1^2+\lambda_2^2$.

Now, From Mariwan's theorem5, we have $|\Psi| = \begin{vmatrix} 6 & 2 \\ 4 & 2 \end{vmatrix} = 4$, $|\Phi| = \begin{vmatrix} 6 & \frac{17}{4} \\ 4 & 1 \end{vmatrix} = -11$ and $\Omega = 2$. Now, we applying equation (19) we get

$$\begin{bmatrix} a_4 \\ a_3 \\ \frac{17}{4} \\ -12 \\ \frac{77}{8} \end{bmatrix} = 2\begin{bmatrix} 0 \\ 0 \\ 1 \\ -(\lambda_1+\lambda_2) \\ \lambda_1\lambda_2 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \\ \frac{3}{2} \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 0 \\ -3 \\ 1 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

Thus, we obtain $\lambda_1\lambda_2 = 5$ and $\lambda_1 + \lambda_2 = 7$. Therefore, $\lambda_1^2 + \lambda_2^2 = (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2 = 39$

Conclusion:

In general, finding oblique asymptote of rational functions needs more mathematical computation whenever numerator and denominator of rational function are long expression, in this case we can use Mariwan's theorem for oblique asymptotes easily and successfully to find oblique asymptotes. Moreover, in this work, the Vieta's formula is generalized and applied to intersection between rational functions and its oblique asymptotes. Also Mariwan's vector equation successfully can be used to find unknown coefficients in rational functions when intersection between rational function and its oblique asymptotes are given.

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